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An Iterative Process for Nonlinear Lipschitzian Strongly Accretive Mappings in L_p Spaces

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Suppose $X=L_p$ (or l_p), $p\geqslant 2$. Let $T: X\to X$ be a *Lipschitzian* and *strongly accretive* map with constant $k\in(0,1)$ and Lipschitz constant L. Define $S: X\to X$ by $Sx=f-Tx-x$. Let $\{C_n\}_{n=1}^{\infty}$ be a real sequence satisfying:

(i) $0<C_n\leqslant k[(p-1)L^2+2k-1]^{-1}$ for each n,

(ii) $\sum_n C_n=\infty$.

Then, for arbitrary $x_0\in X$, the sequence

$$x_{n+1}=(1-C_n)x_n+C_nSX_n, \qquad n\geqslant 0$$

converges strongly to the *unique* solution of $Tx=f$. Moreover, if $C_n=k[(p-1)L^2+2k-1]^{-1}$ for each n, then,

$$\|x_{n+1}-q\|\leqslant \rho^{n/2}\,\|x_1-q\|,$$

where q denotes the solution of $Tx=f$ and

$$\rho=(1-k[(p-1)L^2+2k-1]^{-1})\in(0,1).$$

A related result deals with the iterative approximation of *Lipschitz strongly pseudocontractive* maps in X.

1. Introduction and Preliminaries

Let X be a real normed linear space. A mapping T with domain $D(T)$ and range $R(T)$ in X is called *monotone* [15] if for each x, y in $D(T)$ and *some* real number $t\geqslant 0$, the following inequality is satisfied:

$$\|x-y\|\leqslant\|x-y+t(Tx-Ty)\| \tag{1}$$

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Mappings satisfying (1) for *all* $t \geqslant 0$ are sometimes referred to as *accretive* [2]. If X is a Hilbert space, the accretive condition (1) reduces to

$$\operatorname{Re}\langle Tx - Ty, x - y\rangle \geqslant 0 \tag{2}$$

for all x, y in X. The accretive operators were introduced in 1967 by F. E. Browder [2] and T. Kato [15]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$\frac{du}{dt} + Tu = 0, \qquad u(0) = u_0 \tag{3}$$

is solvable if T is locally Lipschitzian and accretive on X. Browder also proved that if $T: X \to X$ is locally Lipschitzian and accretive then T is *m-accretive*; i.e., the map $(I + T)$, where I denotes the identity map of X, is surjective. This result was subsequently generalized by R. H. Martin [18] to continuous accretive operators. Zarantonello [27] proved that, if H is a Hilbert space, the operator equation

$$x + Tx = h \tag{4}$$

for each $h \in H$ has a unique solution provided T is monotone and Lipschitzian.

For a Banach space X we denote by J the normalized duality map from X to 2^{X^*} given by

$$Jx = \{f^* \in X^* : \|f^*\|^2 = \langle x, f^*\rangle\},$$

where X^* denotes the dual space of X and $\langle\ ,\ \rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single-valued, and if X^* is uniformly convex, then J is uniformly continuous on bounded sets (see, e.g., [3]).

Let X be a Banach space and let K be a nonempty subset of X. A mapping $U: K \to X$ is called *strongly accretive* if for each x, y in K there exists $w \in J(x - y)$ such that

$$\langle Ux - Uy, w\rangle \geqslant k\,\|x - y\|^2$$

for some real constant $k > 0$. Without loss of generality we assume that $k \in (0, 1)$. Strongly accretive mappings are sometimes also called *strictly accretive*. These mappings have been studied by several authors (e.g., [1, 3, 12, 19, 21, 22]). In [19] the following theorem is proved:

THEOREM M (Morales, [19]). *Let X be a Banach space and $T: X \rightarrow X$ be continuous and strongly accretive. Then T maps X onto X.*

An obvious consequence of Theorem M is that for each $f \in X$, the equation $Tx = f$ has a solution in X. We prove (Theorem 1) that a well-known fixed point iteration process converges *strongly* to a solution of this equation. Our method also shows that such a solution must necessarily be unique. A related result (Theorem 2) deals with the iterative approximation of the fixed point of the class of nonlinear Lipschitz *pseudocontractive* mappings (defined below) considered in [7]. We establish a convergence proof which is direct and simpler than the proof given in [7]. Moreover, the method employed here also gives additional information on an explicit error estimate for a particular choice of the iteration parameter.

2. Two Fixed Point Iteration Methods

In this section we describe two fixed point iteration methods given by the following:

(a) *The Ishikawa Iteration Process* (see e.g., [13, 26]) is defined as follows: For K a convex subset of a Banach space X, and T a mapping of K into itself, the sequence $\{x_n\}_{n=1}^{\infty}$ in K is defined by

$$x_0 \in K \tag{5}$$

$$x_{n+1} = (1 - \alpha_n)\, x_n + \alpha_n T y_n \tag{6}$$

$$y_n = (1 - \beta_n)\, x_n + \beta_n T x_n, \qquad n \geqslant 0, \tag{7}$$

where $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ satisfy $0 \leqslant \alpha_n \leqslant \beta_n \leqslant 1$ for all n, $\lim_n \beta_n = 0$, and $\sum_n \alpha_n \beta_n = \infty$; and

(b) *The Mann Iteration Process*, (see e.g., [17, 26]) is similar to the Ishikawa iteration process above but with $\beta_n \equiv 0$, and different conditions placed on α_n. More precisely, with X, K, and x_0 as in part (a), the Mann iteration process is defined by

$$x_0 \in K \tag{8}$$

$$x_{n+1} = (1 - C_n)\, x_n + C_n T x_n, \qquad n \geqslant 0, \tag{9}$$

where $\{C_n\}_{n=0}^{\infty}$ is a real sequence satisfying: $C_0 = 1 : 0 \leqslant C_n < 1$ for all $n \geqslant 1$, and $\sum_n C_n = \infty$. The condition $\sum_n C_n = \infty$ is, in some applications, replaced by $\sum_n C_n(1 - C_n) = \infty$.

The iteration processes described in (a) and (b) above have been studied extensively by several authors for approximating the fixed points of several nonlinear mappings and for approximating solutions of several nonlinear operator equations in Banach spaces (see e.g., [4, 6, 8–12, 17, 19–27]). For a comparison of the two iteration schemes the reader may consult [26].

3. Strong Convergence Theorems for the Solution of $Tx = f$ When T is Lipschitzian and Strongly Accretive

In this section we prove (Theorem 1) that the Mann iteration process converges *strongly* to a solution of the equation $Tx = f$ when T is Lipschitzian and strongly accretive, and $X = L_p$, $p \geqslant 2$. Our method also shows that such a solution must necessarily be unique We need the following preliminaries:

A Banach space X is called *an upper weak parallelogram space* with constant $b \geqslant 0$ if

$$\|x+y\|^2 + b\,\|x-y\|^2 \geqslant 2\,\|x\|^2 + 2\,\|y\|^2 \tag{*}$$

holds for all x, y in X. If L_p has at least two sets of positive finite measure, it is proved in [5] that L_p and l_p $(p \geqslant 2)$ spaces are upper weak parallelogram spaces with $(p-1)$ as the smallest number b such that inequality (*) is satisfied for all x, y in L_p (or l_p), $p \geqslant 2$.

In the sequel we make use of the characterization of upper weak parallelogram spaces in terms of normalized duality mapping given in the following results:

Theorem B (Bynum [5]). *Let X be a Banach space with normalized duality mapping, J. Then X is an upper weak parallelogram space with constant $b \geqslant 0$ if and only if, for each x, y in X and $j \in Jy$,*

$$\|x+y\|^2 \leqslant b\,\|x\|^2 + \|y\|^2 + 2\langle x, j\rangle. \tag{10}$$

Bynum [5] also proved that if $X = L_p$ (or l_p), $p \geqslant 2$, and the single-valued duality map is denoted by j, then for all x, y in X,

$$(p-1)\,\|x+y\|^2 \geqslant \|x\|^2 + \|y\|^2 + 2\langle y, j(x)\rangle.$$

Now, by replacing x by y and y by $(x-y)$ in this inequality we obtain

$$\|x-y\|^2 \leqslant (p-1)\,\|x\|^2 + \|y\|^2 + 2\langle -x, j(y)\rangle.$$

Replace x by $-x$ in the last inequality to obtain,

$$\|x+y\|^2 \leqslant (p-1)\,\|x\|^2 + \|y\|^2 + 2\langle x, j(y)\rangle. \tag{11}$$

We make use of inequality (11) in what follows.

For the remainder of this paper, X denotes an L_p (or l_p) space with $p \geqslant 2$, and the single-valued duality map is denoted by j. The Lipschitz constant of T is denoted by L $(\geqslant 1)$, and the constant appearing in the definition of a strongly accretive map is denoted by $k \in (0, 1)$.

THEOREM 1. *Let $T: X \to X$ be a Lipschitzian and strongly accretive map. Define $S: X \to X$ by $Sx = f - Tx + x$. Let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying:*

(i) $0 < C_n \leqslant k[(p-1)L^2 + 2k - 1]^{-1}$ *for each n,*

(ii) $\sum_n C_n = \infty$.

Then, for arbitrary $x_0 \in X$, the sequence

$$x_{n+1} = (1 - C_n)x_n + C_n Sx_n, \qquad n \geqslant 0$$

converges strongly to the unique solution of $Tx = f$.

Moreover, if $C_n = k[(p-1)L^2 + 2k - 1]^{-1}$ then

$$\|x_{n+1} - q\| \leqslant \rho^{n/2} \|x_1 - q\|,$$

where q denotes the solution of $Tx = f$ and

$$\rho = (1 - k^2[(p-1)L^2 + 2k - 1]^{-1}) \in (0, 1).$$

Proof. The existence of a solution to $Tx = f$ follows from Morales, [9].

Let q be a solution. Observe that S is Lipschitzian with the same Lipschitz constant L, and that q is a fixed point of S. Moreover, for each x, y in X,

$$\begin{aligned} \langle Sx - Sy, j(x-y) \rangle &= -\langle Tx - Ty, j(x-y) \rangle + \|x-y\|^2 \\ &\leqslant -k\|x-y\|^2 + \|x-y\|^2 = (1-k)\|x-y\|^2. \end{aligned}$$

Now, using (11),

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1 - C_n)(x_n - q) + C_n(Sx_n - Sq)\|^2 \\ &\leqslant (p-1)C_n^2\|Sx_n - Sq\|^2 + (1-C_n)^2\|x_n - q\|^2 \\ &\quad + 2C_n(1-C_n)\langle Sx_n - Sq, j(x_n - q) \rangle \\ &\leqslant (p-1)C_n^2L^2\|x_n - q\|^2 + (1-C_n)^2\|x_n - q\|^2 \\ &\quad + 2C_n(1-C_n)(1-k)\|x_n - q\|^2 \\ &= [(1-C_n)^2 + 2C_n(1-C_n)(1-k) + (p-1)C_n^2L^2]\|x_n - q\|^2 \\ &= [(1-C_n)^2 + 2C_n(1-C_n)(1-k) + wC_n^2]\|x_n - q\|^2 \\ &= [1 - 2kC_n + C_n^2(w + 2k - 1)]\|x_n - q\|^2, \end{aligned}$$

where $w = (p-1)L^2$. So, using conditions (i), i.e., $C_n \leqslant k(w+2k-1)^{-1}$, we obtain,

$$\|x_{n+1} - q\|^2 \leqslant (1 - kC_n)\, \|x_n - q\|^2$$
$$\leqslant \exp(-kC_n)\, \|x_n - q\|^2. \tag{12}$$

Iterating the last inequality from $n = 1$ to N we obtain

$$\|x_{N+1} - q\|^2 \leqslant \exp\left(-k \sum_1^N C_n\right) \|x_1 - q\|^2 \to 0 \quad \text{as} \quad N \to \infty$$

since $\sum_n C_n = \infty$. Hence $\{x_n\}_{n=1}^{\infty}$ converges strongly to q.

Uniqueness. Suppose there exists some $q^* \neq q$ which is a solution of $Tx = f$. Then, since q was an arbitrarily chosen solution, repeating the argument of the proof of the theorem relative to q^*, one sees that $\{x_n\}_{n=1}^{\infty}$ also converges to q^*. This contradiction implies $q^* = q$.

Error Estimate. If $C_n = k[(p-1)L^2 + 2k - 1]^{-1}$, for each n, then from inequality (12),

$$\|x_{n+1} - q\|^2 \leqslant (1 - k^2[(p-1)L^2 + 2k - 1]^{-1})\, \|x_n - q\|^2$$
$$\leqslant \leqslant (1 - k^2[(p-1)L^2 + 2k - 1]^{-1})^n\, \|x_1 - q\|^2,$$

so that,

$$\|x_{n+1} - q\| \leqslant \rho^{n/2}\, \|x_1 - q\|,$$

where ρ is as defined, competing the proof of the theorem.

4. Strong Convergence Theorem for the Fixed Point of Lipschitz Strictly Pseudocontractive Mapping in L_p Spaces

We conclude this paper by proving a result which is closely related to Theorem 1. We need the following definition: Let K be a nonempty subset of a Banach space E. A mapping $A: K \to E$ is said to be *strictly pseudocontractive* if there exists $t > 1$ such that the inequality,

$$\|x - y\| \leqslant \|(1+r)(x-y) - rt(Ax - Ay)\| \tag{13}$$

holds for all x, y in K and $r > 0$. It is known that if A is a strict pseudocontraction then $(I - A)$ is strongly accretive with $k = (t-1)/t$ (see e.g., [1] or [7]).

THEOREM 2. *Suppose K is a nonempty closed bounded and convex subset of X and $T: K \to K$ is a Lipschitz strictly pseudocontractive mapping of K into itself. Let $\{C_n\}_{n=0}^{\infty}$ be a real sequence satisfying* (i) $0 < C_n < s[(p-1)L^2 + 2s - 1]^{-1}$ *for each n, where $s = (t-1)/t$ and $t > 1$ is the constant appearing in inequality* (13), *and* (ii) $\sum_n C_n = \infty$. *Then, for arbitrary $x_0 \in K$, the sequences,*

$$\begin{aligned} &x_0 \in K, \\ &x_{n+1} = (1 - C_n)\, x_n + C_n T x_n, \qquad n \geqslant 0 \end{aligned} \tag{14}$$

converge strongly to the unique fixed point of T.

Proof. The existence of a fixed point follows from Deimling, [8].

Let q denote a fixed point of T. Since T is strictly pseudocontractive then $(I - T)$ is strongly accretive. So, for each x, y in K,

$$\mathrm{Re}\langle (I-T)\,x - (I-T)\,y, j(x-y)\rangle \geqslant s\,\|x-y\|^2,$$

where $s = (t-1)/t$. From (14), using (11),

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-C_n)(x_n - q) + C_n(Tx_n - Tq)\|^2 \\ &\leqslant (1-C_n)^2\,\|x_n - q\|^2 + (p-1)\,L^2 C_n^2\,\|x_n - q\|^2 \\ &\quad - 2C_n(1-C_n)\langle Tq - Tx_n, j(x_n - q)\rangle \\ &= (1-C_n)^2\,\|x_n - q\|^2 + (p-1)\,L^2 C_n^2\,\|x_n - q\|^2 \\ &\quad + 2C_{nn}(1-C_n)\,\|x_n - q\|^2 - 2C_n(1-C_n) \\ &\quad \times \langle x_n - Tx_n - q + Tq, j(x_n - q)\rangle \\ &= [(1-C_n)^2 + 2C_n(1-C_n) + (p-1)\,L^2 C_n^2]\,\|x_n - q\|^2 \\ &\quad - 2C_n(1-C_n)\langle (I-T)x_n - (I-T)\,q, j(x_n - q)\rangle \\ &\leqslant [(1-C_n)^2 + 2(1-s)\,C_n(1-C_n) + (p-1)\,L^2 C_n^2]\,\|x_n - q\|^2 \\ &= [(1-C_n)^2 + 2C_n(1-C_n)(1-s) + wC_n^2] \\ &\quad \times \|x_n - q\|^2,\ w = (p-1)L^2 \\ &\leqslant (1 - sC_n)\,\|x_n - q\|^2, \end{aligned}$$

which is exactly inequality (12) with k replaced by s. The rest of the argument now follows exactly as in the proof of Theorem 1, to complete the proof of the theorem.

Remark. Theorem 2 was first proved by the author [7] with different conditions on the real sequence $\{C_n\}_{n=0}^{\infty}$ and with a method different from the one used here. The method used here is direct and significantly simpler than that used in [7]. Moreover, if $C_n = s[(p-1)L^2 + 2s - 1]^{-1}$ for each n, $s \in (0, 1)$, our method here gives the additional information of an explicit error estimate which is obtained as in the proof of Theorem 1.

Problem 1. Is Theorem 3 or 4 extendable to L_p (or l_p) spaces for $1 \leqslant p < 2$?

Problem 2. Can the Ishikawa iteration process be extended to Theorems 3 and 4?

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